

MORE ON TVM, "SIX FUNCTIONS OF A DOLLAR", FINANCIAL MECHANICS

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I want everyone to be very clear on both the "trees" (our basic financial functions) and the "forest" (the idea of the cash flow model). Let's look at the trees in this handout. See also the Appendix in Shilling, pp. 653 ff.

In the handout on present values we learned how to:

- (1) Compute the future value of a one time deposit (compounding).
- (2) Compute the future value of a *series* of periodic payments (an annuity).
- (3) Compute the present value of a one time future receipt (discounting).
- (4) Compute the present value of a series of future receipts.

We will review these, and also how to:

- (5) Determine how much must be set aside periodically to accumulate a certain value (sinking fund).
- (6) Determine the constant periodic payment necessary to repay (amortize) a loan.

Pay particular attention to how each of these "functions of a dollar" are related. We started with a familiar concept, "the miracle of compound interest," and build them up one by one.

Review of the First Four Functions

The key to understanding all this stuff is simply this: a dollar today is worth more than a dollar tomorrow. How much more? That depends on your discount rate.

For each of these functions you have several options open to you:

- (1) Write down the cash flows, figure out the weights for each period and add them up. I prefer to do this in a spreadsheet when I can.
- (2) Shortcut formulas. These can be useful, but excessive focus on "what's the formula" without understanding its derivation can obscure what is actually going on.
- (3) Punch data into your financial calculator. Fast and easy, but as with (2) make sure you understand what goes on behind the scenes.
- (4) Financial tables. These are obsolete, and we no longer teach them. If you're interested, see Shilling's appendix.
- (5) Spreadsheet functions. This is what you'll probably do most often if you become a real estate professional. We illustrate

these methods in class, but cannot rely on them for exams (for obvious reasons).

We go back and forth among methods. Method 1 yields superior understanding.

Function 1: Future Value of a Dollar Today

Suppose we know the present value (a dollar today is worth a dollar today), and we want to find the future value. We showed in the PV handout that the general formula for the future value A of an amount A_0 invested today at rate i to time t is:¹

$$\text{Future Value } FV \equiv A_t = A_0 \cdot (1+i)^t$$

This calculation is easy to carry out with any calculator that calculates exponents, or using the exponent operator ("^") in a spreadsheet.

Function 2: Future Value of a Series of Periodic Payments

By simply extending the formula above to many payments, A ,

$$FV_t = A_1(1+i)^t + A_2(1+i)^{t-1} + \dots + A_t$$

In the special case where each payment A_i is the same, namely A :

$$FV_T = A \sum_{t=1}^T (1+i)^{t-1} = A \frac{(1+i)^T - 1}{i}$$

Again, one can use this formula,² or the financial calculator; one can use the =FV function in a spreadsheet. FV returns the future value of an investment based on periodic, constant payments and a constant interest rate:

$$FV(\text{rate}, \text{nper}, \text{pmt}, \text{pv}, \text{type})$$

Rate is the interest rate per period.

¹ In my handout I wrote A_t as the amount in time t (the future value) and A_0 as the amount in time 0 (today, the present value). Sometimes you will see this written differently, e.g. F or FV for future value and P or PV for present value.

Get used to seeing different notation from different sources. If you rely too much on one set of notation you will be unable to handle future work, since (for better or for worse) everyone uses different symbols.

² Math fans! Step by step derivation of all the important formulas, including the properties of infinite series which we use to derive them, can be found in Stanley Grossman, *Calculus* (Academic Press, 1977), Chapter 6, "Exponentials and Logarithms," especially the sections on applications to economics and finance.

Nper is the total number of payment periods in an annuity.

Pmt is the payment made each period; it cannot change over the life of the annuity. If pmt is omitted, you must include the pv argument.

Pv is the present value, or the lump-sum amount that a series of future payments is worth right now. If pv is omitted, it is assumed to be 0 (zero), and you must include the pmt argument.

Type is the number 0 or 1 and indicates when payments are due. If type is omitted, it is assumed to be 0.

Set "type" equal to 0 if payments are due at the end of the period, and 1 if due at the beginning of the period. (End of period discounting is the norm in real estate).

Function 3: Present Value of a One-Time Receipt

Turning the notion of compounding on its head, if we want to know how much an amount received at time t is worth today (time 0), we showed that by solving Function 1 for A_0 we get

$$PV \equiv A_0 = A_t / (1+i)^t$$

which is the *present value* of future amount A_t . Calculating using the formula is straightforward, in a calculator or in Excel. But be careful; the =PV function in Excel does *not* return the present value of a one time receipt; rather =PV(i,n,A) returns the present value of a *series* of equal receipts A (see Function 4, below).

Of course to calculate the present value of a single given cash flow you may also use the Appendix table, Column 4, "Present Value of 1."

Function 4(a): Present Value of a Series of Future Receipts

To represent PV more generally, we make use of the following notation, where the Greek letter Sigma means "sum" (add over all t periods, from the initial period 0 to the final period T)

$$PV \equiv A_0 = \sum_{t=0}^T \frac{A_t}{(1+i)^t}$$

Spreadsheets and financial calculators will calculate the present values of any set of cash flows you enter in. Be careful if entering a set of flows in a calculator; in an exam (as in a real world problem) it's a good idea to enter the data and calculate the PV twice, as it's easy to mispunch one of a series of cash flows.

Excel's =PV function calculates the net present value of a series of *equal* cash flows: =PV(i,n,A). Excel's =NPV function calculates the net present value of a series of *uneven* (different) cash flows: =NPV(i, range containing the cash flows). A few quirks: Excel (and most other spreadsheets) assume that the cash flow in the first cell in the range is period 1, the second cell period 2, etc. This implies that if you have a period 0 (undiscounted) cash flow you must calculate it as per the following example:

=C12+NPV(.1,D12:F12), where the period 0 cash flow is in cell C12 and cells D12, E12 and F12 contain periods 1, 2 and 3 respectively, and these are to be discounted at 10 percent.

Function 4(b): Present Value of A Series of Equal Payments

When each amount A_t is the same, then we have the present value of an annuity (see Function 2 for the future value of same). If we can compute the future value we can certainly discount it back to the present, i.e. by computing:

$$PV = A \frac{1 - (1+i)^{-n}}{i}$$

Since present values are additive, Function 4(b) can be handy even when some flows are different; one can compute the present value in parts, if one is careful to keep track to avoid double counting. Personally, I prefer to use a financial calculator and/or a spreadsheet without the shortcut to lessen the chance of error.

Alternatively you can use the =PV function in Excel.

Function 4(c): Present Value of a Perpetuity

A shortcut: using calculus it can be shown that, in the limit, if each and every A_t is the same (let's say A), as T approaches infinity this formula becomes:

$$A_0 = \frac{A}{i}$$

which is easy and requires no table or fancy calculator.

Two New Functions

Function 5: Sinking Fund Accumulation

How much will have to be set aside each period in an account paying given interest to accumulate a particular future value? This would be very useful to you if you ever have to save for a particular purpose (e.g. to pay your children's college tuition!). In appraisal we will use this to compute the so-called "recapture premium" required for a "bottom up" cap rate (of which more later).

From Function 2, we know how to compute the future value of a series of payments:

$$FV = A_1(1+i)^t + A_2(1+i)^{t-1} + \dots + A_t$$

but this is a simple problem where (1) we assume each A_i is the same (equal payments), call the amount A , and (2) we know the future value instead of the

payments. The solution is straightforward:

$$FV = A * [(1+i)^t + (1+i)^{t-1} + \dots + 1]$$

by rearranging terms we get

$$A = FV * S$$

where S is the "sinking fund factor" we need, i.e.

$$S = \frac{1}{(1+i)^t + (1+i)^{t-1} + \dots + 1} = \frac{i}{(1+i)^t - 1}$$

S can also be found using a financial calculator. One can also find the required payment directly using the =PMT function in Excel, using the form =PMT(i,n,0,FV) where i and n are as before; 0 is entered as the third element because this element is the present value of the expression (which is zero in this case), and FV is the desired future value.

Function Six: The Constant Periodic Payment Required to Amortize a Loan

What are the payments required to fully amortize (pay off principal as well as interest) a particular loan amount, given a constant interest rate and a fixed term?

Rethink the problem as follows. Each period the lender receives an equal payment, which he or she discounts to the present (see function 4). The interest rate is, in this case, the rate at which we discount. Our problem then becomes turning PV on its head: Given PV and an interest rate at which to discount, what series of equal annual payments is equivalent?

Example. What annual payments are required to amortize an 8 percent, 3 year, \$5,000 loan?

$$\begin{aligned} 5000 &= A/1.08 + A/1.08^2 + A/1.08^3 \\ &= A [1/1.08 + 1/1.08^2 + 1/1.08^3] \\ &= A * 2.5771 \end{aligned}$$

$$\begin{aligned} \text{So } A &= 5000/2.5771 \\ &= 5000 * .3880 \\ &= \$1940.17 \end{aligned}$$

Generalization. With a bunch of manipulation, it is not too hard to solve for the payment A which yields a present value PV,

$$A = PV * \frac{1}{\sum_{t=1}^T \frac{1}{(1+i)^t}} = PV * \frac{i}{1 - \frac{1}{(1+i)^T}}$$

where i is the interest rate and T is the term of the loan. One can use the formula; or any business calculator has such a function built in directly. In Excel use the =PMT function: =PMT(i,n,PV).

Tips on Applying These Six Functions

One of the most common mistakes is to confuse monthly and annual amounts in calculations or tables. It's easy enough to calculate a monthly mortgage payment and multiply by 12 to get annual payments on a monthly mortgage; but it's also easy enough to forget (I still do so occasionally).

A second common mistake is to forget to add *positive* cash flows and subtract *negative* ones. Sounds hard to forget, but you'd be surprised.

Another common mistake is to forget to remember that spreadsheet formulas assume discounting of the first cash flow when in many problems the first flow is undiscounted (see above, discussion of =NPV).

Students often initially confuse the =PMT, =PV and =NPV functions in Excel and/or the equivalent functions in their financial calculators. Spend a little time practicing with them if you are using spreadsheets. That suggests one thing the appendix tables are good for: see if you can replicate numbers in the Appendix tables, e.g. the payment needed to amortize a \$1 loan. This is good practice whether you use spreadsheets, financial calculators, or formulas.

Mortgage Balance

What is the outstanding balance on a mortgage? Answering this question is easy if we realize that this is just the present value of the *remaining* stream of interest payments, discounted at the rate of interest.

For example, suppose we had a \$100,000 30 year fixed rate mortgage at 12 percent (monthly payments). Monthly payments would be \$1,028.61 (check). What is the outstanding balance after 5 years?

since
there are
300

$$MB = 1028.61 * \sum_{t=1}^{300} \frac{1}{(1.01)^t} = 1028.61 * 94.9465 = \$97,662.97$$

payments left after 5 years. After 5 years, the balance has declined by only \$2,337.03.

There are other ways to calculate the balance. Sometimes I prefer to calculate the interest and payment of each mortgage payment separately and explicitly, if I don't have too many to do. Excel has functions =CUMIPM and =CUMPRINC which compute the portion of a payment that is interest and principal, respectively. Printed amortization schedules are also widely available.