UNDERSTANDING PRESENT VALUES AND THE TIME VALUE OF MONEY

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Understanding present value is the key to understanding valuation and appraisal of real estate, real estate finance and taxation, and in fact almost every topic from here on in. It is quite general and applicable to all business decisions. Understanding present values is the key to understanding the difference between stocks and flows, wealth and income. Present values are the basis for making investment decisions, project appraisal, and cost-benefit analysis. All of finance is based on this model.[1]

The key to understanding **Present Values** is simply this: A dollar today is worth more than a dollar tomorrow. How much more? That depends on your *discount rate*. While they are related (and can happen to be the same in some contexts), the "discount rate" is not quite the same as the "rate of interest." This will become clear shortly.

For a moment, we will *not* differentiate between the rate of interest and the discount rate. We will study a simple example when they are the same. Then we will discuss conditions under which they can be different.

The Time Value of Money

Given a choice, would you rather have a dollar today, or a dollar a year from now?[2] Each of us would rather have the dollar today; if we wanted to spend it today, we could, and if we would rather spend it in the future, we could put in in the bank, draw interest for a year, and in one year have the dollar *and* the interest. If the rate of interest were (for example) 10 percent, a year from now we would have our dollar and an additional 10 cents.[3] If we held it for two years we would have $1.10 after the first year, and since 10 percent of $1.10 is 11 cents, we would have $1.21 -- the extra penny, the interest on the previous period's interest, is what savings and loans call "the miracle of compound interest" in their promotional materials. As we will see, compounding may not seem like a big deal for low rates of interest and short time periods, but if you are in for the long haul it makes a lot of difference.

If we hold the dollar for three years, year-by-year we accumulate:

```
Today   Year 1                  Year 2                  Year 3
1.00    (1.00*.1)+1.00 = 1.10   (1.10*.1)+1.10 = 1.21   (1.21*.1)+1.21 = 1.33
```

Without compounding, we would only have 1.30. If the rate of interest was 20% we would accumulate:

```
Today   Year 1                  Year 2                  Year 3
```

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[1]   If you want to learn more, consult a textbook on corporate finance, such as Brealey and Myers, *Principles of Corporate Finance*.

[2]   Assume that a dollar is not a trivial amount of money.

[3]   In these examples I assume rates of interest that are compounded once a year. Quarterly, monthly or even continuous compounding changes the numerical results a little, but all the ideas are the same.

1.00   (1.00*.2)+1.00 = 1.20   (1.20*.2)+1.20 = 1.44   (1.44*.2)+1.44 = 1.73

Without compounding we would only have 1.60 (we have 13/60=22% more interest because of compounding!)

<u>A Little Algebra</u>

Can you see that to get the present value of year 3 directly, we can substitute successively backwards?

Notice that (e.g.) (1.44*.2)+1.44 is the same as 1.44*(1+.2)

or symbolically, for any amount "A,"

$$A_t = A_{t-1}*(1+i)$$

but this also implies

$$A_{t-1} = A_{t-2}*(1+i)$$

So it must also be true that:

$$A_t = A_{t-2}*(1+i)*(1+i).$$

We can keep substituting as long as we like, going one period back every time, so:

$$A_t = A_{t-3}*(1+i)*(1+i)*(1+i)$$

and so on. So if our period t was the third period, t-3 would be period zero, or:

$$A_3 = A_0*(1+i)*(1+i)*(1+i)$$

$$= A_0*(1+i)^3$$

or in our example,

$$A_3 = 1.00*(1.2)^3 = 1.00*1.73 = 1.73$$

as in our example from above. Now you can see that the general formula for the final value A of an amount A_0 invested today at rate i to time t is:

$$A_t = A_0*(1+i)^t$$

If you followed this little bit of algebra the first time through, great. If not, make up some simple examples yourself, with particular values of A_0 and i, and see that you get the same result from the formula that you do from doing it step by step. For example, suppose that A_0=$1 and i=10%, and that we hold the investment for 3 years. From the formula we get

$$A_3 = 1*(1.1)^3 = 1*(1.33) = \$1.33$$

which is exactly what we got for our step by step example above!

<u>The Discount Rate</u>

The discount rate is just a measure of how much we value that dollar today versus the dollar a year from now. In a sense, the interest rate is the discount rate of the financial markets, in the aggregate.

If we could all borrow as much as we needed at "the" rate of interest -- and lend any extra we have in any period at the same rate[4] -- if there were no differences in risk, all economic activities yielded a single market rate of return, there were no transactions costs of borrowing, no liquidity constraints, and all agents had perfect information about all markets, "the" rate of interest would be "the" discount rate. But such is not the case, and discount rates differ.

What is *your* discount rate? We can answer that easily enough if you answer the following kinds of questions:

> *If you had a choice between a dollar today and $1.10 a year from now, would you take the dollar today or the $1.10?*

If you answered the dollar today, we know your discount rate is at least 10 percent, because you gave up the chance to make an extra 10 percent. Perhaps you can get 10 percent or more in a bank somewhere, or you have an investment in mind that you think will yield more than 10 percent, or perhaps you want to consume today enough that you are willing to give up consuming the same amount plus 10 percent a year from now.[5]

> *If you had a choice between a dollar today and $1.50 a year from now, would you take the dollar today or the $1.50?*

If you answered that you prefer $1.50 a year from now, we have established that your discount rate is less than 50 percent. We can continue our line of questioning until we reach a point where we get something like the following:

> *If you had a choice between a dollar today and $1.19 a year from now, would you take the dollar today or the $1.19?*

You answer $1 today, so your discount rate is above 19 percent.

> *If you had a choice between a dollar today and $1.21 a year from now, would you take the dollar today or the $1.21?*

You answer $1.21, so your discount rate is less than 21 percent.

> *If you had a choice between a dollar today and $1.20 a year from now, would you take the dollar today or the $1.20?*

Your answer is: "I don't know!" You are indifferent between a dollar today and $1.20 a year from now. Your discount rate is 20 percent. You would be indifferent between $1 today and -- how much? -- *three* years from now?[6]

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[4]   When you deposit money in the bank, you lend the bank money.

[5]   Economists often talk about "the tradeoff between present and future consumption."

[6]   Answer: $1.73, of course.

Present Values
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What have we established so far?

(1)   The value of a dollar depends on when we receive it. The timing
       of cash flows is important.

(2)   The discount rate is a measure of the time value of money, i.e.
       how we trade off present versus future cash flows.[7]

(3)   We can relate this more precisely with the formula $A_t = A_0(1+i)^t$,
       which tells us how much an amount A_0 is worth at the end of t
       periods.

Let us take this fundamental equation and turn it on its head. If we want to
know how much an amount received at time t is worth today (time 0), we can
just do the following. We know that

$A_t = A_0(1+i)^t$, so

$A_t/A_0 = (1+i)^t$, and therefore

$A_0 = A_t/(1+i)^t$

which is the *present value* of A_t!

In many practical problems we have more than one cash flow. For example, a
bond yields interest over several periods. You can simply add present
values. For example, the present value of two payments, at times 1 and 2, is
simply:

$A_0 = A_1/(1+i)^1 + A_2/(1+i)^2$

and if you have, say, 10 payments at 10 periods, you have 10 such terms. To
represent this more generally, we make use of the following notation, where
the greek letter Sigma means "sum" (add over all t periods, from the initial
period 0 to the final period T)

$$A_0 = \sum_{t=0}^{T} \frac{A_t}{(1+i)^t}$$

Example: Suppose someone gave you $100 today (period 0) and for each of
three succeeding periods (years 1-3). Suppose your discount rate was 10
percent. We can lay this out as follows:

Period:	0	1	2	3	Arithmetic Sum

Cash Flow:	100	100	100	100	400
	----	----	----	----	
Discount Factor:	1.00	1.10	1.21	1.33	

Present
Value

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[7]   Recall the important if somewhat subtle point made above that the discount
rate is related to but is not quite the same thing as the rate of interest.

Discounted					-----
Flow:	100	91	83	75	349

While the arithmetic sum is obviously $400, note that $100 a year from now is only worth $91 today. (Loosely speaking, perhaps if you had $91 you could put it in the bank for a year and get $100). $100 two years from now is worth $83 today (notice $83 today is worth $83(1.1)(1.1)=$83(1.1)^2=$100). And $75(1.1)^3=$100, and so on. The sum of the *discounted* cash flows is the present value, in this case, $349.

Suppose the discount rate *increases*. Then the PV of future payments *decreases*. Suppose the discount rate was 20 percent. Then

Period:	0	1	2	3
Cash Flow:	100	100	100	100
Discount Factor:	1.00	1.20	1.44	1.73

					Present Value
Discounted					-----
Flow:	100	83	69	58	311

By the way, this is the key to understanding the relationship between interest rates and the price of bonds that you discussed in principles of economics: if the market rate of interest goes up, the price of bonds previously issued goes down.

A Digression on Bonds

Most bonds start paying one period in the future (not in period 0).[8] Consider a four year bond which is a promise to pay $100 for four years. Suppose the interest rate on the day of issue is 10 percent:

Period:	1	2	3	4
Cash Flow	100	100	100	100
Discount:	1.10	1.21	1.33	1.46

				Present Value
Discounted				-------
Flow:	91	83	75	68	317

The *face value* or *par value* of the bond -- the amount actually written on the

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[8]   There are many kinds of bonds. We analyze a simple version of the most common kind of bond. All other bonds can be analyzed using the same general framework, although the details will differ.

face -- is $317.[9] You pay $317 and get a promise to be paid $400 over 4
years.

What if *after* the bonds were printed but before they were sold the interest
rate went up?

```
    Discount Rate:     20.0%

    Period:      1     2     3     4

    Cash Flow  100   100   100   100
               ----  ----  ----  ----
    Discount: 1.20  1.44  1.73  2.07

                                        Present
                                         Value
    Discounted                          -------
    Flow:        83    69    58    48       259
```

Now, even though the face value says $317, buyers will only pay $259. The
bond *trades* at a *discount* from par. If the interest rate goes up, the price
of bonds goes down. In completely analogous fashion, if the interest rate
goes down the price of bonds goes up.

Note that many bonds are bought and sold *after* they are issued. To price the
above bond if the interest rate is (say) 20 percent and the first two
payments have been made, just add the present value of the remaining payments
($58+$48).

Can we think of an income property as a kind of bond?

Perpetuities

Another useful trick for valuing a long term bond is as follows. Look at the
following 10 year bond:

```
  Discount
  Rate:     20.0%

  Period:      1     2     3     4     5     6     7     8     9     10

  Cash Flow  100   100   100   100   100   100   100   100   100   100
             ----  ----  ----  ----  ----  ----  ----  ----  ----  ----
  Discount: 1.20  1.44  1.73  2.07  2.49  2.99  3.58  4.30  5.16  6.19

                                                                      Present
                                                                       Value
  Discounted                                                          -----
  Flow:        83    69    58    48    40    33    28    23    19    16    419
```

If we calculated the same PV for a 20-year bond, the PV would be $496. For a

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30-year bond, PV would be $499. You could quickly convince yourself that as the holding period of a bond gets longer, the PV of a constant flow R at a discount rate i approaches:

PV = R/i

In fact, recall our formula for valuation of discrete payments:

$$A_0 = \sum_{t=0}^{T} \frac{A_t}{(1+i)^t}$$

It can be shown that, in the limit, if each and every A_t is the same, as t approaches infinity this formula becomes:

$$A_0 = \frac{A}{i}$$

Are there any such investments? Bonds which promise to pay forever are called perpetuities or consols. They are not common in this country but have been issued by the British government.[10] But many forms of real estate, particularly *land*, last a very long time. In real estate, the simple formula was used a lot before spreadsheets, when valuation using a single year's net operating income and a single "cap rate" (capitalization, or discount rate) was computationally convenient. It's still used as a shortcut method for initial "quick and dirty" analysis.

It is worth emphasizing that an infinitely long lived investment still has a finite value. This brings us back to the idea of a limit, as in calculus. The figure to the right illustrates the case of a bond or a house or whatever that returns $100 per year for 100 years. The arithmetic sum of those returns, represented by the area under the rectangle, is 100*100=$10,000. Suppose our discount rate is 10 percent. The height of the curve labled "Discounted" at each year is the present value of that year's $100 payment. The total present value, represented by the area under the curve, from the formula for a

Cash Flows and Discounted Cash Flows $100 Per Year for 100 Years

Discount Rate: 10 Percent

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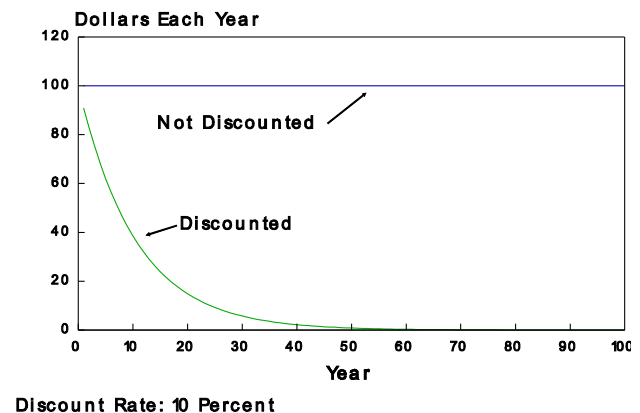
[10]   Back in the days when the sun never set on their empire. Issuing perpetuities may well have had something to do with a fundamental attitude or world view the British had at that time, but we'll leave further comment to psycho-historians.

discrete present value, is $999.93.[11] How good an approximation is the
continuous formula? Using the simpler formula, PV = 100/.1 = $1,000. While
this is not an investment of infinite life, the simple formula gives a pretty
good approximation. What's really different is the difference between the
arithmetic sum and the other two (by a factor of 10!)

A Simple Application to Real Estate

The method has many other applications: pricing any asset or investment
(bond, stock, real estate, whatever). And note there's no reason cash flows
have to be the same every year, or even positive.

Suppose Donald Frump invests $300 million dollars to buy a building that pays
him $100 million for three years (and then collapses). If his discount rate
is 10 percent, The Donald sets up the following model:

Discount Rate:	10.0%				
Period:	0	1	2	3	
Cash Flow:	-300	100	100	100	
	----	----	----	----	
Discount:	1.00	1.10	1.21	1.33	
					Present Value

Discounted Flow:	-300	91	83	75	-51

Notice that his $300 m. outlay only returns $249 in PV terms in the
succeeding years. The present value rule for an investment is simple: only
undertake investments that yield *positive* net present values.

An alternative rule that is sometimes discussed is the target "internal rate
of return." The internal rate of return is just the discount rate that
happens to make the present value of such a set of cash flows equal zero.
While there is a subtle difference between the two rules,[12] they are in a
sense two sides of the same coin. Other rules, like "payback period" rules,
are inferior.

Generally, it is difficult to solve analytically for the internal rate of
return. IRR's are usually calculated by iteration. The steps are as
follows.

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[11]   I won't show the calculation here for lack of space, but it's easy to do
in a spreadsheet. Try it as an exercise. See me if you have trouble or get a
different answer.

[12]   Can you see that the present value depends on the size of the project,
and tells you how much your net worth increases if you undertake it? If you
could choose only one, would you rather undertake a small project with a higher
rate of return, or a larger one with a slightly smaller return but a larger
present value? What other information might you need to answer this question?
(Hint: what is the capital market like? Can you borrow if you need to to
undertake the larger project?)

      (1)   Make an initial guess of the IRR, and calculate the net present value;

      (2)   If the initial NPV is positive, (assuming negative cash flows followed by positive), then the initial guess of IRR is too low (i.e. you are putting too much weight on the later, positive cash flows).

      (3)   Increase your guess by a little bit, and recalculate the NPV. Keep doing this until the NPV just switches from positive to negative. If you've used small increments (say 0.001) to change the guess, you are at least within that increment of the true IRR.

If your original guess produced an NPV that was negative, and your initial cash flows are negative, followed by positive cash flows, subtract some small increment from your initial guess until the NPV turns positive, and again you're done.

This is actually how financial calculators and spreadsheets calculate IRRs. The advantage they have is that they are fast. (There are a few tricks to writing a good algorithm to calculate IRR, that will be reviewed in Real Estate 631 when you write your own program to calculate IRRs in VBA).

Another point we'll discuss in some detail later, but which should be mentioned from the outset, is that if cash flows change sign more than once, there will be more than one IRR. (For example, if you spend money today, receive positive net rents for a few years, then make a big capital improvement in year 3, receive positive rents for a few more years and then sell the building, there will be two IRRs. In general, each time cash flow signs "flip," another IRR rears its head).

Spreadsheet Implementation

Excel has many financial functions, but here are three you will need to know like the back of your own hand.

To calculate the present value of a stream of equal payments,

```
=PV(rate, nper, pmt)
```

where rate is the discount rate, nper is the number of periods, and pmt is the periodic (monthly/annual) payment. Note that pmt is a single number (because it is repeated nper times).

To calculate the net present value of a set of cash flows of different amounts (and signs),

```
=NPV(rate, range of values)
```

where rate is again the discount rate, and "range of values" contains each and every cash flow. "Range of cash flows" can be entered as a set of numbers directly, or can refer to a range of cells. If the latter, the cash flows need to be lined up in adjoining cells (in a row or a column) and each cell to the right or down must represent the next period.

To calculate the internal rate of return:

```
=IRR(range of values, guess)
```

where "range of values" contains the cash flows, and "guess" contains a guess of the IRR.

Some Excel functions have optional inputs. For example, the PV function actually has two more inputs:

```
=PV(rate, nper, pmt, [fv], [type of discounting])
```

where fv is some future value (lump sum at the end of the periods; assumed zero if not entered) and "type of discounting" is whether the cash flows are assumed to arrive at the beginning or end of the period (e.g. January 1 or December 31 for an annual cash flow). If omitted, or the value "0" entered, Excel assumes end-of-period discounting. (This is the most common assumption in business problems). If the value 1 is entered, payments are assumed to occur at the beginning of the period.

The IRR function assumes a "guess" of 0 if no number is entered. Whether an explicit guess is entered or not, IRR is calculated iteratively, and Excel limits the number of iterations. Thus, if the "guess" (or 0, if not entered) is far away from the true IRR, Excel may not reach it, and you will receive a result "#NUM". If this occurs, try another guess, perhaps moving IRR 0.2 or so up or down as needed until the function converges.

Also, note that the NPV function begins discounting IMMEDIATELY. For example, if you use NPV (incorrectly) to calculate our simple example above,

```
=NPV(0.1,-300,100,100,100)
```

returns $46.65, which while close, is incorrect. The correct answer can be found by entering:

```
=-300+NPV(100,100,100)
```

which returns the correct answer, $51.31.

Reversion, or Salvage Value

So far, in both the bond examples and the real estate example, we have only dealt with two kinds of cases: a few cash flows, and an infinite number (an investment that lasts forever). Surely you can see that we can easily add the proceeds from sale at some period as an additional cash flow. This is called the "reversion," or sometimes "the residual," or "salvage value."

In light of this, can you see the flaw in the common statement, "I'm not concerned about (dividends, rents, interest, or other periodic payments), I'm in it for capital gains?" A capital gain is an increase in value at some future period. The key to understanding capital gains is to ask what, in turn, determines value at that future period? *The (expected) dividends, rents or interest forward of that future period!* What does this tell us about stocks which have low dividend payouts relative to their current stock prices (and/or high price-to-earnings ratios)?

Expectations

Another subtle but important point: when we talk about future rents, or future dividends, or future interest payments -- how do we know the future? The mechanics of valuation are simple, and appear "scientific," or objective, but the necessary inputs -- expectations about the future -- are subjective.

In the case of bonds, the nominal payment is known with certainty.[13] On the other hand no one *knows* future dividends or rents. But we do have an *expectation* -- an estimate or guess -- of what they will be. We can think of the market value of real estate as being determined by "the market's expectation" of future rents or other payments. When your expectation exceeds the market, you would tend to buy. When your expectation is that the market is optimistic, you would tend to sell. But an interesting question is: can the market be wrong systematically, or much of the time? Or is the market "efficient?" Most evidence suggests that in the long run, and on average, U.S. financial markets markets do value assets fairly efficiently.[14] Real estate markets are much less efficient.

This concludes our *introduction* to a fascinating topic. Other questions that we will address later in the course include applying these ideas to valuation of real estate, investment analysis, risk, the role of taxes and subsidies, real versus nominal cash flows, and portfolio allocation.

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[13]   Assuming for the moment that there is no risk of default or late payment.

[14]   An obvious exception is when one market participant has information that others don't. Then you can certainly beat the market. But there are other risks. Ask Ivan Boesky, or (more recently, if for smaller $$$, Martha Stewart).